Strategies for comparing decimal numbers with the same whole-number part

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The strategies employed by 130 Grade 5 Brisbane students in comparing decimal numbers were investigated. Nine different strategies were identified, some of which indicated sophisticated understanding while others indicated restricted understanding. Most students had a predominant strategy which determined success or failure on particular items. Predomonant strategy distibution was compared with that of French, USA and Israeli students.

Students have difficulties in acquiring an understanding of decimal numbers (e.g., Behr, Harel, Post & Lesh, 1992; Bigelow, Davis & Hunting, 1989; Kieren, 1988; Resnick et al., 1989; Wearne & Hiebert, 1988). These difficulties appear to stem from impoverished understandings of whole-number numeration (e.g., Hiebert & Wearne, 1985; Resnick et al., 1989; Wearne & Hiebert, 1988) and the notion of a fraction (e.g., Bezuk, 1988; Resnick et al., 1989). A discussion of these difficulties and their underlying causes is provided in Baturo and Cooper (1995).

Comparing decimal numbers which have the same whole-number part requires both an understanding of place value and the fraction concept. Thus it is a particularly rich topic in which to study students' strengths and weaknesses in these domains. Sackur-Grisvard and Leonard (1985) found that half of the Grades 4 and 5 French children tested generally used the three systematic (predominant) but incorrect strategies described below to decide which was the greater of two decimal numbers with the same whole-number part. For mnemonic ease, Resnick et al. (1989), named these strategies as *whole-number, zero* and *fraction* rules.

Whole-number rule. The decimal number with more decimal places (i.e., the "longer") is the larger. For example, 4.156 would be considered larger than 4.7 because 4.156 has three decimal places whilst 4.7 has only one. That is, the decimal fractions are treated as whole numbers in which 156 is larger than 7 and this behaviour is thought to stem from an overgeneralisation of an impoverished method for comparing whole numbers, namely, "the number with the most digits is the largest".

Zero rule. The decimal number with one or two zeros to the immediate right of the decimal point is the smaller. For example, 4.09 is correctly considered as smaller than 4.8. The zero rule, therefore, always produces a correct result but for an inappropriate reason. This rule is most often invoked by students who predominantly use the whole-number rule and is thus seen as a special case of the whole-number rule.

Fraction rule. The decimal number with the fewer decimal places (i.e., the "shorter") is the larger. For example, 4.2 would be considered to be larger than 4.865 because 4.2 has one decimal place and 4.865 has three. Resnick et al. (1989) argued that this rule probably stems from an overgeneralisation of the principle for comparing common fractions, namely, "the larger the denominator, the smaller the fraction".

It should be noted that each of these three rules will produce correct comparisons in particular instances but the *expert rule* only will produce correct comparisons in all situations. Although this rule was used as a category in their study, Resnick et al. did not describe it. There appeared to be a tacit understanding that experts compared the digits in like places from left to right.

As a result of analysing the incorrect strategies revealed in the Sackur-Grisvard and Leonard study (1985), Resnick et al. hypothesised that the timing of the introduction of decimal-fraction and common-fraction recording in various curricula would affect the predominant strategy used. Therefore, they predicted that USA and Israeli students would invoke the fraction rule more often than the French students because USA and Israeli curricula introduce common-fraction recording long before decimal-fraction recording whereas in France, the reverse schedule is adopted. For the same reason, Resnick et al. expected the French students to exhibit the zero rule more frequently. However, it was expected that all three groups of students would exhibit predominantly the whole-number rule because all students are familiar with whole numbers before being introduced to fractions per se. The results of their study would appear to support their predictions. (See Table 2 in the Results section.)

This paper reports on a study of the comparison strategies employed by 130 Grade 5 Brisbane students when comparing decimal numbers. The aims of the study were: (1) to identify comparing strategies used by Brisbane students; (2) to relate strategy usage to item type; and (3) to compare the predominant strategy use of the Brisbane students' with that of the USA, French and Israeli students.

Method

Sample

Three classes of Grade 5 students from each of two large Brisbane state schools were selected for the study, making 130 students altogether. The schools were selected to represent different socioeconomic backgrounds (one middle-to-high, the other middle-to low).

Instrument

The instrument comprised nine comparison items (see Table 1) which were chosen to reveal Sackur-Grisvard and Leonard's (1985) strategies. When read across, Table 1 shows the expected success rate if a particular strategy is used predominantly by each student. When read down, Table 1 shows that the successful comparison of some items can result from both correct and incorrect strategies (e.g., Items E and H).

Table 1: Items used and the expected success rate for a predominant strategy.

Rule	*Items*								
	A	*B*	*C*	*D*	*E*	*F*	*G*	*H*	*I*
	8.6	*5.24*	*4.2*	*0.5*	*4.7*	*0.2*	*9.64*	*0.04*	*6*
	2.8	*3.79*	*4.63*	*0.36*	*4.08*	*0.10*	*9.5*	*0.4*	*3.7*
EX	*3*	*3*	*3*	*3*	*3*	*3*	*3*	*3*	*3*
WN	*3*	*3*	*7*	*7*	*7*	*7*	*3*	*7*	*7*
ZE	*3*	*3*	*N/A*	*N/A*	*3*	*N/A*	*N/A*	*3*	*N/A*
FR	*3*	*3*	*7*	*3*	*3*	*3*	*7*	*3*	*3*

Note. EX = expert; WN = whole number; ZE = zero; FR = fraction.

The numbers were presented to minimise place value problems and were restricted to tenths and hundredths because of the year level of the students.

Items A, B and I had different whole-number parts. Items A and B were used to establish whether the students could invoke a correct comparison technique when different whole numbers were involved. It was expected that most students would complete these items successfully. Item I was included as a nonprototypical type that would most likely challenge the students who had developed correct comparison rules which were not conceptually-based.

Of the items that incorporated decimal numbers in which the whole-number part was the same (C, D, E, F, G, H), Items C, D, F and G provided two correct and two incorrect applications for both the whole-number and fraction strategies, while Items E and H provided two opportunities for zero strategies to be exhibited.

Procedure

This instrument was given to each class as a pencil-and-paper test. Each child was then interviewed individually and asked two questions: (a) *Have you selected the larger number in this item?* (b) *How can you tell?*

The first question was designed to establish whether there had been a momentary lapse in concentration, resulting in the selection of the smaller number instead of the larger one. The students could change their answers if they wished but were required to justify their second choice and state why they thought their first choice was incorrect. The strategies used by the students were encoded during the interviews. The interviewer was free to probe the students' responses.

Results

Individual strategies per item

The students' responses to each item were classified into categories of comparison strategies. The initial categorisation replicated those of Resnick et al. (1989), namely, EX, WN, ZE, FR and OTHER. However, it soon became apparent that other categories were required. For example, in Item C (4.2 and 4.63), several students used a renaming strategy in which the 2 tenths was renamed as 20 hundredths and then 20 was compared with 63. Similarly, in Item E (4.7 and 4.08), just as many students ignored the zero (and, therefore, 4.7<4.08) as used the zero rule (and, therefore, 4.7>4.08).

Ultimately, five other strategies emerged from the Brisbane study – *renaming, benchmarking, zero-ignored, fraction-inverted* and *expert backwards*. (See Table 2 for summary of the students' correct and incorrect responses and the strategies they used in comparing the numbers within each item.) Whilst renaming and benchmarking were alluded to by Resnick et al. in their report, these strategies were not included in their final categorisation.

The *renaming* (RE) strategy, exhibited by 14.6% of the Brisbane students, consisted of "equalising" the fractions by renaming tenths as hundredths.

The *benchmarking* (BE) strategy, exhibited by 9.2% of the Brisbane students, was based on estimation. For example, in Item D, 0.5 was seen as a half and 36 hundredths was seen as less than a half. It was used most successfully by 9 students in Item H where 0.4 was compared with 0.04.

The *zero-ignored* (ZI) strategy, exhibited by 15.4%, emerged in the two items that were designed to assess whether students used the zero rule (Items E and H). In Item E, students using this strategy considered that 4.08 was larger than 4.7 because the 8 in the 4.08 was larger than 7 in the 4.7.

The *expert-backward* (EB) strategy, exhibited by 2,3% only, consisted of comparing like places from right to left, not left to right.

The *fraction-inverted* (FI) rule, exhibited by 4.6%, involved comparing the number of hundredths required to make one whole with the number of tenths required to make one whole thus failing to consider the size of a tenth compared to the size of a hundredth. This behaviour occurred most often in Items G and H where the students explained that "hundredths are worth more than tenths". This reasoning led to success in Item G but to failure in Item H.

Table 2: Categorisation of the strategies used in each item by the Brisbane students.

Items		%	Correct usage of strategies (n = 130)	Incorrect
A	8.6, 2.8	98.5	EX (125); BE (2); WN (1)	EB (2)
B	5.24, 3.79	93.1	EX (120); DI (1)	EB (8)
C	4.2, 4.63	81.5	EX (49); RE (11); WN (44); UN (2)	WN (1); FR (20); UN (2)
D	0.5, 0.36	58.5	EX (42); RE (15); WN (3)*, FR (13); BE (2); UN (2),	WN (50); BE (1); FI (1); EB (1)
E	4.7, 4.08	66.2	EX (51); RE (12); FR (13); ZE (10)	WN (26); FI (1); ZI (17)
F	0.2, 0.10	50.0	EX (40); RE (14); FR (7); BE (3), UN (1)	WN (62); BE (1); FI (1)
G	9.64, 9.5	90.8	EX (50); RE (8); WN (53); FR (5); BE (1); UN (1)	FR (11); FI (2); RE (1); UN (1)
H	0.04, 0.4	85.4	EX (54); RE (9); FR (22), BE (9); ZE (14); ZI (2); UN (1)	WN (21); FI (4); ZI (6)
I	6, 3.7	72.3	EX (93); RE, (1);	WN (34); EB (1); RE (1)*

Note. EX = *expert;* WN = *whole number;* FR = *fraction;* ZE = *zero;* ZI = *zero-ignored;* RE = *renaming;* EB = *expert-backwards;* FI = *fraction inverted;* BE = *benchmarking;* UN = *unclassified.*

Table 3 shows the use of strategies on those items in which the whole-number parts of the given decimal numbers were the same.

Table 3 Comparison of strategies used in Items C-H (same whole numbers).

Items		Per cent usage of strategies across Items C to H								
		EX	WN	FR	ZE	ZI	RE	EB	FI	BE
C	4.2, 4.63	37.7	34.6	15.4	N/A	N/A	10.0	00.7	00.0	00.0
D	0.5, 0.36	32.3	40.7	10.0	N/A	N/A	11.5	00.7	01.5	02.3
F	0.2, 0.10	30.7	47.7	05.5	N/A	N/A	10.7	00.7	00.7	02.3
G	9.64, 9.5	38.5	40.7	12.3	N/A	N/A	06.9	00.0	01.5	00.7
	Mean (%)	34.8	40.9	10.7	N/A	N/A	09.8	00.6	00.9	01.3
E	4.7, 4.08	39.2	20.0	10.0	07.7	13.1	09.2	00.0	00.7	00.0
H	0.04, 0.4	41.5	16.1	16.9	10.7	06.1	06.9	00.0	03.1	06.9
	Mean (%)	40.3	18.0	13.4	09.2	09.6	08.1	00.0	01.9	03.4

Predominant strategies across items

In order to determine whether the students used a predominant strategy, their response across all relevant items were considered. Responses were identified as predominant in the following manner.

ZE, ZI: Restricted to Items E and H and must be used in both items.

EX, RE, WN, FR, BE, FI, EB: Used in the majority of the six items (C-H).

Table 4 compares the Brisbane students' predominant comparison strategies with those exhibited by the French, USA and Israeli students.

Table 4 Results of the Brisbane study and Resnick et al.'s (1989) study of the predominant comparison strategies employed by middle-school students.

Country	*Per cent using strategy predominantly*									
	EX	*RE*	*EB*	*WN*	*ZE*	*ZI*	*FR*	*BE*	*FI*	*UN*
USA (Gr 6 - 17)	*18*	*N/A*	*N/A*	*35*	*0*	*N/A*	*18*	*N/A*	*N/A*	*29*
Israel (Gr 5 -21)	*19*	*N/A*	*N/A*	*19*	*14*	*N/A*	*33*	*N/A*	*N/A*	*14*
France (Gr 4 - 37)	*30*	*N/A*	*N/A*	*41*	*11*	*N/A*	*8*	*N/A*	*N/A*	*11*
(Gr 5 - 38)	*53*	*N/A*	*N/A*	*18*	*24*	*N/A*	*3*	*N/A*	*N/A*	*3*
Brisbane (Gr 5 - 130)	*34*	*7*	*0*	*25*	*8*	*3*	*6*	*0*	*0*	*17*

Within the group classified as "other" in Table 4, approximately 15% used two predominant strategies interchangeably – EX/WN (7%), EX/FR (5%), RE/F (1.5%), WN/RE (0.75%), and WN/FR (0.75%). The remaining 2% could not be classified as the students appeared to use a variety of strategies.

Discussion and conclusions

The nine comparison strategies used by the Brisbane students included the four strategies from Sackur-Grisvard and Leonard (1985) and Resnick et al. (1989) – *expert, whole number, fraction* and *zero* – and five extra strategies, namely, *renaming, expert backwards, zero-ignored, benchmarking* and *fraction-inverted*. The most commonly used strategies were *expert* and *whole number*.

The *renaming* strategy was expected because Brisbane teachers commonly teach students to "equalise decimal lengths" by adding zeros to the end of decimal numbers when comparing and operating decimal numbers. The *zero-ignored* strategy was also expected because of the known problems students have when working with numbers that have internal zeros. However, the *benchmarking* strategy was unexpected and suggests that it may be the result of increased attention to number sense in Brisbane schools.

Some strategies reflected sophisticated understanding of decimal numbers whilst others reflected a restricted understanding. Sophisticated understanding was evident in the *benchmarking* strategy. For example, when comparing the numbers in Item G (9.64 and 9.5), *benchmarking* students chose 9.64 as the larger number because 64 hundredths was closer to 1 than was 5 tenths. Restricted understanding was exhibited by students who used the *zero-ignored* strategy in Items E and H. These students ignored the zero in 4.08 (Item E) thus considering it to be 4.8. This strategy led to difficulties with item H as students then considered 0.4 and 0.04 to be equal. This conflict caused another strategy to be invoked, most commonly the *whole-number* strategy. However, 6 students maintained that the numbers were equal.

For some strategies, it was difficult to determine the quality of student understanding. For example, the interviews revealed that for some students the *renaming* strategy was a conceptually-based rule as they were able to explain that the 2 tenths in 4.2 (Item C) had the same value as 20 hundredths which was smaller than the 63 hundredths in 4.63. However, some other students were invoking the strategy because "the teacher said to add an imaginary zero" and thus the rule was not conceptually based. Similarly, it is highly likely that some of those students who consistently use the *expert* strategy do so without any understanding of the additive feature of the place value system (Resnick et al., 1989).

As revealed in Tables 2 and 3, performance varied according to item type in both strategy and correctness. When the decimal numbers to be compared had the same fraction (i.e, the "same length"), the expert strategy prevailed. When the decimal numbers had different fractions (i.e., "different lengths"), other strategies were invoked with success as predicted in Table 1. The *expert* strategy was used 94.2% for Items A and B but reduced to 53.3% across the nine items. Across Items C to I, this number fell to 41.6% whilst for those items which had the same whole number (Items C-H), the rate fell to 36.6%.

Apart from a few careless errors, the students who used the *expert, renaming,* and *benchmarking* strategies were always successful. The *whole-number* and *fraction* strategies were successful only if the item supported their use and this never occurred for both together. The *zero-ignored* and *expert-backwards* strategies led to incorrect results.

Item F had the lowest success rate (50.0%). It had been assumed that the students would employ the *whole-number* strategy here rather than the *zero* because of the 10 hundredths. The number 10 is the most embedded numbers within the decimal number system and is the first new place that exposes young children to the place value notion. Even those students who normally used the more successful expert and renaming strategies reverted to more inappropriate strategies in this instance. Items D and E had the next lowest success rates. Item I revealed the dilemma for students when faced with a nonprototypical example and also had one of the lower success rates. Surprisingly, Item H had a much higher success rate than Item E (both "zero" items possibly because the zero ones supported the *fraction* strategy and mitigated against the *whole-number* and *zero* strategies.

Predominant strategies across items

As shown in Table 4, the Brisbane students' systematic usage of strategies appears to be similar to the French results from Resnick et al. (1989) – high use of the *expert* and *whole-number* strategies (particularly if *renaming* is combined with *expert*), reasonable use of the *zero* strategy (particularly if *zero-ignored* is combined with *zero*), and low use of the *fraction* strategy. Like France and unlike USA and Israel, the mathematics curricula used in Brisbane supports the teaching of decimal notation before common fraction notation. This appears to support the argument of Resnick et al. that the reasons for predominant comparison-strategy distribution lie with the mathematics curricula of the countries being compared.

The unclassified students mostly used two strategies. In many cases, this dual strategy use indicated that the student was in transition to a more correct strategy use. Therefore, the data from this study could allow development of comparison expertise to be sequenced in terms of strategy use and understanding. Understanding the sequence involved in comparing decimal numbers will enable predominant strategies to be the basis of focused remedial interventions.

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